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Construction of Fermion Mass Matrices Yielding Two Popular Neutrino Scenarios

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Abstract

A new procedure proposed recently enables one to start from the quark and lepton mass and mixing data at the low scale and construct mass matrices which exhibit simple SO(10) structure at the SUSY GUT scale. We elaborate here on the numerical details which led us to an SO(10) model for the quark and lepton mass matrices that explain the known quark data at the low scale along with the observed depletions of solar- and atmospheric-neutrinos. We also apply the procedure to a second scenario incorporating the solar-neutrino depletion and a 7 eV tau-neutrino for the cocktail model of mixed dark matter but find the SO(10) model deduced in this case does not exhibit as simple a structure as that observed for the first scenario.

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I. GENERAL APPROACH AND APPLICATION

In a recent letter,[1] the authors sketched a new approach which enables one to make use of quark and lepton mass and mixing data at the low energy scale to construct an SO(10)-symmetric fermion mass matrix model at the supersymmetric grand unified scale from which the low energy results can be derived. This "bottom-up approach" should be contrasted with the usual procedure, where one introduces some ansatz for the fermion mass matrices at the grand unification scale, from which certain predictions can be made at the low scale. In proposing such an ansatz, one must take care not to violate any of the known data at the low scale. This conventional method has a very extensive literature dating from the early work of Fritzsch[2] up to the present models proposed by many authors[3] in SO(10) SUSY GUTS, where the evolution of the Yukawa couplings from the SUSY GUT scale to the weak scale plays a major role. In some of the most recent work along these lines, authors have attempted to impose[4] as many texture zeros as possible (typically five or six) for the up and down quark matrices, or considered[5] just one **10** and one **126** Higgs representations of SO(10) contributing to the mass matrices, and then deduced the consequences of these assumptions. In doing so, they find that the combined tau-neutrino mass and mixing angles do not accurately fit either the popular cocktail model[6] of mixed dark matter or the oscillation explanation of the observed atmospheric depletion[7] of muon-neutrinos.

In our approach, on the other hand, one must input all the known or presumed known masses and mixings in order to construct numerically the mass matrices by a method proposed in the quark context by Kusenko.[8] Since the neutrino mass and mixing data are not well known at this time, many scenarios can be considered for the starting point. In the letter cited above, we illustrated the procedure with neutrino data extracted from the non-adiabatic Mikheyev-Smirnov-Wolfenstein[9] (MSW) interpretation of the observed solar electron-neutrino depletion[10] and from the muon-neutrino and tau-neutrino mixing interpretation of the observed atmospheric muon-neutrino depletion effect.[7] Here we shall elaborate on the numerical details which led to the mass matrices in the SO(10) framework proposed in Ref. 1 and apply the same technique to a second neutrino scenario involving the same solar electron-neutrino depletion, but now in the presence of a 7 eV tau-neutrino which provides 30% of the missing dark matter, the rest arising from the supersymmetric neutralinos in the cocktail model[6] of mixed dark matter. In this new approach, we can identify what assumptions regarding texture zeros and minimal Higgs content must be relaxed in order to obtain accurate model fits to the two scenarios in question.

We first restate here the basic steps for the construction of the quark and lepton mass matrices in the new approach:

- Start from the known and/or presumed-known quark and lepton masses, m_q's, m_ℓ's and m_ν's; and quark and lepton mixing matrices, V_{CKM} and V_{LEPT}, at the low scales.

- Evolve the masses and mixing matrices to the SUSY GUT scale using the appropriate renormalization group equations (RGEs) for the minimal supersymmetric standard model (MSSM).

- Construct complex symmetric M^U, M^D, M^E, and $M^{N_{eff}}$ matrices for the up and down quarks, charged leptons and light neutrinos using a modified procedure of Kusenko[8] described later. Two parameters x_q and x_ℓ allow one to adjust the diagonal/off-diagonal nature of the quark and lepton mass matrices.

- Vary x_q and x_ℓ systematically over their support regions while searching for as many pure **10** or pure **126** $SO(10)$ contributions to the matrix elements as possible.

- For the "best" choice of x_q and x_ℓ, construct a simple model of the mass matrices with as many texture zeros as possible.

- Evolve the mass eigenvalues and mixing matrices determined from the model at the SUSY GUT scale to the low scale and compare the results with the starting input data.

In Sect. II we shall assign values to the quark and lepton masses and mixings for the two scenarios investigated in this paper. Evolution to the SUSY GUT scale is discussed in Sect. III. Numerical construction of the mass matrices is explained in Sect. IV, followed by the SO(10) model constructions in Sect. V. There we also compute the quark and lepton mass eigenvalues and mixing matrices in the two models and evolve the results downward to the low scales to compare the SO(10) model results with the original quark and lepton input parameters. In Sect. VI we draw our conclusions.

II. MASSES AND MIXINGS AT THE LOW SCALES

Uncertainties in the quark masses and mixings lie within relatively broad bounds. The light quark mass ratios are quite accurately determined from current algebra, while the absolute values are more uncertain.[11] This is especially true for the strange quark mass. We shall adopt as input the central values at 1 GeV quoted by Gasser and Leutwyler[12] about ten years ago, which are as good as any obtained since then. The c and b quark masses are specified at their running mass scales, while the corresponding top quark mass is much less certain[13] since its discovery has yet to be made. With $m_t^{phys} \sim 160$ GeV, we adopt as starting input the following quark masses[12]

$$\begin{aligned} m_u(1\mathrm{GeV}) &= 5.1 \text{ MeV}, & m_d(1\mathrm{GeV}) &= 8.9 \text{ MeV} \\ m_c(m_c) &= 1.27 \text{ GeV}, & m_s(1\mathrm{GeV}) &= 175 \text{ MeV} \\ m_t(m_t) &= 150 \text{ GeV}, & m_b(m_b) &\simeq 4.25 \text{ GeV} \end{aligned} \tag{2.1a}$$

The Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix,[14] on the other hand, is becoming better known with time, its main uncertainties[15] being V_{cb}, and especially the $|V_{ub}/V_{cb}|$ ratio and the CP-violating phase. We adopt at the weak scale the central values

$$V_{CKM} = \begin{pmatrix} 0.9753 & 0.2210 & (-0.283 - 0.126i) \times 10^{-2} \\ -0.2206 & 0.9744 & 0.0430 \\ 0.0112 - 0.0012i & -0.0412 - 0.0003i & 0.9991 \end{pmatrix} \qquad (2.1b)$$

where we have assumed a value of 0.043 for V_{cb} and applied strict unitarity to determine V_{ub}, V_{td} and V_{ts}.

In contrast, although the charged lepton masses are precisely known, the lepton mixings and neutrino masses remain uncertain. But a reasonable starting point has emerged with the increased knowledge gained from the solar neutrino experiments[10] involving the chlorine experiments of Davis, the water Cerenkov experiments of the Irvine-Michigan-Brookhaven and the Kamiokande collaborations and the recent gallium experiments of the SAGE and GALLEX collaborations. Taken together, the depleted electron-neutrino fluxes observed compared with the standard solar model predictions suggest that non-adiabatic MSW resonant conversion[9] of the electron-neutrinos into muon-neutrinos in the solar interior is most likely responsible. The central values deduced for this effect are $\delta m^2_{12} \sim 5 \times 10^{-6}$ eV^2 and $\sin^2 2\theta_{12} \sim 8 \times 10^{-3}$.

With regard to the tau-neutrino mass and mixings, two scenarios are popular and conflicting, if one does not assume near degeneracy of the neutrino masses or the existence of a new sterile neutrino. We shall make no such assumption. In the first scenario labeled (A), one suggests that muon-neutrinos oscillate into tau-neutrinos on their passage through the atmosphere and hence deplete the flux of muon-neutrinos relative to electron-neutrinos to explain the observed atmospheric depletion.[16] The central values for this interpretion are: $\delta m^2_{23} \sim 2 \times 10^{-2}$ eV^2 and $\sin^2 2\theta_{23} \sim 0.5$.

Alternatively, in the second scenario labeled (B) and popularly known as the cocktail model,[6] one speculates that tau-neutrinos account for 30% of the dark matter as a source of missing hot dark matter, while supersymmetric neutralinos serve as a source of cold dark matter and account for the remaining 70%. For this case, the simplest interpretation is that the tau-neutrino has a mass of 7 eV. The present accelerator data on $\nu_\mu - \nu_\tau$ oscillations then place an upper limit of $\sin^2 2\theta_{23} \lesssim 10^{-3}$ on the mixing angle.[17]

We take for the lepton input in neutrino scenario (A)

$$\begin{aligned} m_{\nu_e} &= 0.5 \times 10^{-6} \text{ eV}, & m_e &= 0.511 \text{ MeV} \\ m_{\nu_\mu} &= 0.224 \times 10^{-2} \text{ eV}, & m_\mu &= 105.3 \text{ MeV} \\ m_{\nu_\tau} &= 0.141 \text{ eV}, & m_\tau &= 1.777 \text{ GeV} \end{aligned} \tag{2.2a}$$

and

$$V^{(A)}_{LEPT} = \begin{pmatrix} 0.9990 & 0.0447 & (-0.690 - 0.310i) \times 10^{-2} \\ -0.0381 - 0.0010i & 0.9233 & 0.3821 \\ 0.0223 - 0.0030i & -0.3814 & 0.9241 \end{pmatrix} \tag{2.2b}$$

We have simply assumed a value of for the electron-neutrino mass to which our analysis is not very sensitive and constructed the lepton mixing matrix[18] by making use of the unitarity conditions with the same phase in (2.1b) and (2.2b). For scenario (B), we use

$$\begin{aligned} m_{\nu_e} &= 0.5 \times 10^{-6} \text{ eV}, & m_e &= 0.511 \text{ MeV} \\ m_{\nu_\mu} &= 0.224 \times 10^{-2} \text{ eV}, & m_\mu &= 105.3 \text{ MeV} \\ m_{\nu_\tau} &= 7.0 \text{ eV}, & m_\tau &= 1.777 \text{ GeV} \end{aligned} \tag{2.3a}$$

and

$$V^{(B)}_{LEPT} = \begin{pmatrix} 0.9990 & 0.0447 & (-0.289 - 0.129i) \times 10^{-2} \\ -0.0446 & 0.9989 & 0.0158 \\ 0.0036 - 0.0013i & -0.0157 - 0.0001i & 0.9998 \end{pmatrix} \tag{2.3b}$$

In scenario (A) the tau-neutrino mass is of the order of 0.1 eV, while the 23 element of the leptonic mixing matrix is large, while in scenario (B) the tau-neutrino mass is fifty times larger, but the 23 element of the leptonic mixing matrix is very small; in fact, this mixing matrix is very close to the identity.

III. EVOLUTION TO THE SUSY GUT SCALE

We now evolve the low energy data to the SUSY GUT scale, where any simplicity due to the SO(10) symmetry should apply. In order to use analytic expressions for the running variables, we shall use the one-loop renormalization group equations (RGEs) with numbers taken from the work of Naculich.[19] The supersymmetry breaking scale is assumed to lie at $\mu_{SUSY} = 170$ GeV, while the GUT scale, where the gauge couplings are unified, occurs at $\bar{\mu} = 1.2 \times 10^{16}$ GeV.

The connection between the running mass m_α of a fermion and its corresponding Yukawa coupling y_α is defined by

$$m_\alpha = y_\alpha(v/\sqrt{2}) \begin{cases} \sin\beta, & \alpha = u, c, t, \nu_e, \nu_\mu, \nu_\tau \\ \cos\beta, & \alpha = d, s, b, e, \mu, \tau \end{cases} \tag{3.1a}$$

where $\tan\beta = v_u/v_d$ is the ratio of the up quark to the down quark VEVs and $v = 246$ GeV, the electroweak symmetry-breaking scale. The Yukawa coupling running between the 1 GeV scale for the light quarks and leptons or the running mass scale for the heavy quarks and the supersymmetry breaking scale μ_{SUSY} is governed by the gauge couplings and can be summarized in terms of ratios η_α of the couplings at the low scale to those at μ_{SUSY}. We shall use

$$\begin{aligned} \eta_u &= \eta_d = 2.17, \quad \eta_s = 2.16 \\ \eta_c &= 1.89, \quad \eta_t = 1.00, \quad \eta_b = 1.47 \\ \eta_{\nu_e} &= \eta_{\nu_\mu} = \eta_{\nu_\tau} = 1.03 \\ \eta_e &= \eta_\mu = 1.03, \quad \eta_\tau = 1.02 \end{aligned} \tag{3.1b}$$

Only the third family quark and charged lepton Yukawa couplings are assumed[19] to contribute to the nonlinear part of the Yukawa coupling evolution from the supersymmetry breaking scale to the GUT scale. In this approximation one finds that the μ_{SUSY} scale couplings are given in terms of the GUT scale Yukawa couplings by

$$\begin{aligned} y_u(\mu_{SUSY}) &= \bar{y}_u A_u B_t^3, & y_d(\mu_{SUSY}) &= \bar{y}_d A_d B_b^3 B_\tau \\ y_c(\mu_{SUSY}) &= \bar{y}_c A_u B_t^3, & y_s(\mu_{SUSY}) &= \bar{y}_s A_d B_b^3 B_\tau \\ y_t(\mu_{SUSY}) &= \bar{y}_t A_u B_t^6 B_b, & y_b(\mu_{SUSY}) &= \bar{y}_b A_d B_b^6 B_\tau B_t \\ y_{\nu_e}(\mu_{SUSY}) &= \bar{y}_{\nu_e} A_n, & y_e(\mu_{SUSY}) &= \bar{y}_e A_e B_b^3 B_\tau \\ y_{\nu_\mu}(\mu_{SUSY}) &= \bar{y}_{\nu_\mu} A_n, & y_\mu(\mu_{SUSY}) &= \bar{y}_\mu A_e B_b^3 B_\tau \\ y_{\nu_\tau}(\mu_{SUSY}) &= \bar{y}_{\nu_\tau} A_n, & y_\tau(\mu_{SUSY}) &= \bar{y}_\tau A_e B_b^3 B_\tau^4 \end{aligned} \tag{3.2a}$$

where the gauge evolution factors A_α are equal to

$$\begin{aligned} A_u &= 3.21, \quad & A_d &= 3.13 \\ A_n &= 1.37, \quad & A_e &= 1.48 \end{aligned} \tag{3.2b}$$

and the Yukawa evolution factors are approximately equal to

$$\begin{aligned} B_t &\simeq \left[1 + \bar{y}_t^2 K_u\right]^{-1/12}, \quad & K_u &= 8.65 \\ B_b &\simeq \left[1 + \bar{y}_b^2 K_d\right]^{-1/12}, \quad & K_d &= 8.33 \\ B_\tau &\simeq \left[1 + \bar{y}_\tau^2 K_e\right]^{-1/12}, \quad & K_e &= 3.77 \end{aligned} \tag{3.2c}$$

By combining (3.1) and (3.2), we can find the Yukawa couplings at the grand unification scale. In doing so, we adjust $m_b(m_b)$ and $\tan\beta$ so that complete Yukawa unification[20] is achieved at $\bar{\mu}$, i.e., $\bar{m}_\tau = \bar{m}_b = \bar{m}_t/\tan\beta$. This is accomplished by choosing $m_b(m_b) = 4.09$ GeV at the running b quark mass scale[21] and $\tan\beta = 48.9$. The evolved masses for the quarks at $\bar{\mu}$ are then found to be

$$\begin{aligned} \bar{m}_u &= 1.098 \text{ MeV}, & \bar{m}_d &= 2.127 \text{ MeV} \\ \bar{m}_c &= 0.314 \text{ GeV}, & \bar{m}_s &= 42.02 \text{ MeV} \\ \bar{m}_t &= 120.3 \text{ GeV}, & \bar{m}_b &= 2.464 \text{ GeV} \end{aligned} \tag{3.3a}$$

and for the leptons in scenario (A)

$$\begin{aligned} \bar{m}_{\nu_e} &= 0.581 \times 10^{-6} \text{ eV}, & \bar{m}_e &= 0.543 \text{ MeV} \\ \bar{m}_{\nu_\mu} &= 0.260 \times 10^{-2} \text{ eV}, & \bar{m}_\mu &= 111.9 \text{ MeV} \\ \bar{m}_{\nu_\tau} &= 0.164 \text{ eV}, & \bar{m}_\tau &= 2.464 \text{ GeV} \end{aligned} \tag{3.3b}$$

Since the third family terms control the Yukawa couplings in the RGEs, only the following V_{CKM} and V_{LEPT} mixing matrix elements evolve in leading order and result in

$$\begin{aligned} \bar{V}_{ub} &= (-0.2163 - 0.0963i) \times 10^{-2}, & \bar{V}_{13} &= (-0.634 - 0.285i) \times 10^{-2} \\ \bar{V}_{cb} &= 0.0329 & \bar{V}_{23} &= 0.3508 \\ \bar{V}_{td} &= 0.0086 - 0.0009i & \bar{V}_{31} &= 0.0205 - 0.0028i \\ \bar{V}_{ts} &= -0.0315 - 0.0002i & \bar{V}_{32} &= -0.3502 \end{aligned} \tag{3.3c}$$

while the other mixing matrix elements receive smaller corrections which can be neglected; however, in doing so the unitarity of the mixing matrices is not quite preserved.

In scenario (B), $\bar{m}_{\nu_\tau}$ should be replaced by 8.127 eV, and the lepton mixing matrix elements in (3.3c) by

$$\begin{aligned} \bar{V}_{13} &= (-0.265 - 0.118i) \times 10^{-2} & \bar{V}_{23} &= 0.0145 \\ \bar{V}_{31} &= 0.0033 - 0.0012i & \bar{V}_{32} &= -0.0144 - 0.0001i \end{aligned} \tag{3.3d}$$

IV. NUMERICAL CONSTRUCTION OF MASS MATRICES

Having found the masses and mixing matrices at the GUT scale, we can now construct numerically the quark and lepton mass matrices by making use of a procedure suggested by Kusenko[8] for the quark mass matrices. Since the quark mixing matrix V_{CKM} of the charged-current couplings in the mass bases is unitary and represents an element of the unitary group U(3), one can express it in terms of one Hermitian generator of the corresponding U(3) Lie algebra times a phase parameter α by writing

$$V_{CKM} = U'_L U^\dagger_L = \exp(i\alpha H) \tag{4.1a}$$

where

$$i\alpha H = \sum_{k=1}^{3} (\log v_k) \frac{\prod_{i\neq k}(V_{CKM} - v_i I)}{\prod_{j\neq k}(v_k - v_j)} \tag{4.1b}$$

in terms of the eigenvalues v_j of V_{CKM} by making use of Sylvester's theorem.[22] The transformation matrices from the weak to the mass bases are given in terms of the same generator but modified phase parameters such that

$$U'_L = \exp(i\alpha H x_q), \qquad U_L = \exp\left[i\alpha H(x_q - 1)\right] \tag{4.2}$$

and relation (4.1a) is preserved.

The quark mass matrices in the weak basis are then related to those in the diagonal mass basis by

$$M^U = U'^\dagger_L D^U U'_R, \qquad M^D = U^\dagger_L D^D U_R \tag{4.3a}$$

where D^U and D^D are the diagonal matrices in the mass bases with entries taken from (3.3a). In what follows, we shall be interested in constructing quark and lepton mass matrices in the SO(10) framework which are complex symmetric. This requires that only the **10** and **126** irreducible representations of SO(10) develop vacuum expectation values, while the antisymmetric **120** does not. In the higher grand unified groups exhibiting family symmetry such as SO(14) or SO(18), the complex symmetric representations are naturally selected. If we impose this restriction on (4.3a), we can eliminate the transformation matrices for the right-handed fields in favor of

$$M^U = U_L^{\prime\dagger} D^U U_L^{\prime\dagger T}, \qquad M^D = U_L^{\dagger} D^D U_L^{\dagger T} \tag{4.3b}$$

The parameter x_q then controls the diagonal/off-diagonal nature of the mass matrices, where the up quark mass matrix is diagonal for $x_q = 0$, while the down quark mass matrix is diagonal for $x_q = 1$. It suffices to expand V_{CKM}, $U_L^{\prime}$ and U_L to third order in α in order to obtain accurate expressions for the mass matrices M^U and M^D.

A similar argument can be applied in order to construct the light neutrino and charged lepton mass matrices, $M^{N_{eff}}$ and M^E, from the lepton masses and V_{LEPT} mixing matrix. Here the generator and phase parameter are different as V_{CKM} and its eigenvalues v_i are replaced by V_{LEPT} and $v_i^{\prime}$, but relations similar to (4.1) through (4.3) still obtain with x_ℓ replacing x_q.

In order to complete the construction of the mass matrices, we must select pairs of values for x_q and x_ℓ lying in the unit square support region, $0 \leq x_q, x_\ell \leq 1$. For this purpose, we search for a simple SO(10) structure for the mass matrices. In the SO(10) framework, the renormalizable Yukawa interaction Lagrangian for the non-supersymmetric fermions is given by

$$\mathcal{L}_Y = -\sum_i \bar{\psi}^{c(16)} f^{(10_i)} \psi^{(16)} \phi^{(10_i)} - \sum_j \bar{\psi}^{c(16)} f^{(126_j)} \psi^{(16)} \bar{\phi}^{(126_j)} + \text{h.c.} \tag{4.4a}$$

where the f's represent Yukawa coupling matrices. For fermions in the fundamental **16** representation, the only Higgs fields allowed lie in the symmetric **10** and **126** representations and the antisymmetric **120**. As is customary, we ignore the latter, so the mass matrices are complex symmetric and given by

$$\begin{aligned} M^U &= \sum_i f^{(10_i)} v_{ui} + \sum_j f^{(126_j)} w_{uj} \\ M^D &= \sum_i f^{(10_i)} v_{di} + \sum_j f^{(126_j)} w_{dj} \\ M^{N_{Dirac}} &= \sum_i f^{(10_i)} v_{ui} - 3\sum_j f^{(126_j)} w_{uj} \\ M^E &= \sum_i f^{(10_i)} v_{di} - 3\sum_j f^{(126_j)} w_{dj} \end{aligned} \tag{4.4b}$$

where v_{ui} and w_{uj} are the **10** and **126** VEV contributions to the up quark and Dirac neutrino matrices, and similarly for the down quark and charged lepton contributions. The equations in (4.4b) can be inverted to determine the sum of the **10** and sum of the **126** contributions separately. At this stage we do not know how many **10** and **126** representations of each type are necessary.

By varying the x_q and x_ℓ parameters over the unit square support region and by allowing all possible signs to appear in the diagonal matrix entries of D^U, D^D, D^E and $D^{N_{eff}}$, we can search for a set of mass matrices which have either pure **10** or pure **126** structure for as many matrix elements as possible. Such a preferred choice is found for scenario (A) with $x_q = 0$ and $x_\ell = 0.88$ for which the mass matrices are constructed to be

$$M^U = diag(-.1098 \times 10^{-2},\ 0.3140,\ 120.3) \tag{4.5a}$$

$$M^D = \begin{pmatrix} (-.8847 + .1072i) \times 10^{-4} & (-.9688 - .0080i) \times 10^{-2} & (-.4967 - .2371i) \times 10^{-2} \\ (-.9688 - .0080i) \times 10^{-2} & -.3705 \times 10^{-1} & (.8221 + .0001i) \times 10^{-1} \\ (-.4967 - .2371i) \times 10^{-2} & (.8221 + .0001i) \times 10^{-1} & 2.460 \end{pmatrix} \tag{4.5b}$$

$$M^E = \begin{pmatrix} (-.5339 + .0027i) \times 10^{-3} & (.4135 - .0425i) \times 10^{-3} & (-.4005 - .0837i) \times 10^{-2} \\ (.4135 - .0425i) \times 10^{-3} & 0.1160 & 0.1020 \\ (-.4005 - .0837i) \times 10^{-2} & 0.1020 & 2.453 \end{pmatrix} \tag{4.5c}$$

in units of GeV and

$$M^{N_{eff}} = \begin{pmatrix} (.4839 + .1534i) \times 10^{-4} & (-.9059 - .1304i) \times 10^{-3} & (.3023 + .0374i) \times 10^{-2} \\ (-.9059 - .1304i) \times 10^{-3} & (.1465 - .0001i) \times 10^{-1} & (-.5065 + .0002i) \times 10^{-1} \\ (.3023 + .0374i) \times 10^{-2} & (-.5065 + .0002i) \times 10^{-1} & 0.1502 \end{pmatrix} \tag{4.5d}$$

in units of electron volts.

For scenario (B), the simplest SO(10) construct, with as many pure **10** or pure **126** matrix elements as possible, is obtained with $x_q = 0.5$ and $x_\ell = 0$, but further investigation reveals there is only one texture zero. In place of that, we examine a slightly more complicated choice which parallels that of scenario (A) with four texture zeros for which $x_q = 0$ and $x_\ell = 0.3$. The numerical matrices in this case are exactly the same for the quarks as given in (4.5a,b), while the lepton matrices are replaced by

$$M^E = \begin{pmatrix} (-.4256 + .0079i) \times 10^{-3} & (.3469 - .0021i) \times 10^{-2} & (-.4781 - .2042i) \times 10^{-2} \\ (.3469 - .0021i) \times 10^{-2} & 0.1120 & 0.2386 \times 10^{-1} \\ (-.4781 - .2042i) \times 10^{-2} & 0.2386 \times 10^{-1} & 2.463 \end{pmatrix} \tag{4.6a}$$

in units of GeV and

$$M^{N_{eff}} = \begin{pmatrix} (.5994 + .5325i) \times 10^{-5} & (.0240 - .1243i) \times 10^{-4} & (.7495 + .2887i) \times 10^{-2} \\ (.0240 - .1243i) \times 10^{-4} & -.2448 \times 10^{-2} & (-.3518 + .0008i) \times 10^{-1} \\ (.7495 + .2887i) \times 10^{-2} & (-.3518 + .0008i) \times 10^{-1} & 8.127 \end{pmatrix} \tag{4.6b}$$

in units of electron volts.

V. IDENTIFICATION OF SO(10) MODELS AND PREDICTIONS OF THE MODELS

A. Neutrino Scenario (A) with Atmospheric Neutrino Depletion

In order to construct an SO(10) model which closely approximates the numerical matrices found in (4.5), we take note of the following features. The up quark matrix M^U is diagonal and the structure for M^D and M^E is approximately given by

$$M^D \sim M^E \sim \begin{pmatrix} 10,126 & 10,126 & 10 \\ 10,126 & 126 & 10 \\ 10 & 10 & 10 \end{pmatrix} \tag{5.1a}$$

as observed from Eqs. (4.4b), with M^D_{11}, M^E_{12} and M^E_{21} anomalously small, i.e., smaller than expected when compared to the pattern for the other elements. We shall, in fact, assume that these elements exhibit texture zeros.[4] Most of the other elements are essentially real with the 13 and 31 elements of M^D and M^E the major exceptions. Hence we let only the latter elements be complex.[23] If we also assume that the same **10** and **126** VEVs contribute, respectively, to the 33 and 22 diagonal elements of M^U and M^D, we find

$$M^U \sim M^{N_{Dirac}} \sim diag(10, 126;\ 126;\ 10) \tag{5.1b}$$

If we now seek as simple a structure as possible for the four matrices, we are led numerically to the following choices for the Yukawa coupling matrices at the GUT scale

$$\begin{aligned} f^{(10)} &= diag(0,\ 0,\ f^{(10)}_{33}), & f^{(126)} &= diag(f^{(126)}_{11},\ f^{(126)}_{22},\ 0) \\ f^{(10')} &= \begin{pmatrix} f^{(10')}_{11} & f^{(10')}_{12} & f^{(10')}_{13} \\ f^{(10')}_{12} & 0 & f^{(10')}_{23} \\ f^{(10')}_{13} & f^{(10')}_{23} & 0 \end{pmatrix}, & f^{(126')} &= \begin{pmatrix} 0 & f^{(126')}_{12} & 0 \\ f^{(126')}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.2}$$

The model requires a minimum of two **10**'s and two **126**'s of SO(10) with **10**$'$ and **126**$'$ having no VEVs in the up direction. There are four texture zeros in the M^U and M^D matrices taken together. The four mass matrices are then given by

$$M^U = f^{(10)}v_u + f^{(126)}w_u \tag{5.3a}$$

$$M^{N_{Dirac}} = f^{(10)}v_u - 3f^{(126)}w_u \tag{5.3b}$$

$$M^D = f^{(10)}v_d + f^{(126)}w_d + f^{(10')}v'_d + f^{(126')}w'_d \tag{5.3c}$$

$$M^E = f^{(10)}v_d - 3f^{(126)}w_d + f^{(10')}v'_d - 3f^{(126')}w'_d \tag{5.3d}$$

and assume the simple textures

$$\begin{aligned} M^U &= diag(F',\ E',\ C') \qquad & M^{N_{Dirac}} &= diag(-3F',\ -3E',\ C') \\ M^D &= \begin{pmatrix} 0 & A & D \\ A & E & B \\ D & B & C \end{pmatrix} \qquad & M^E &= \begin{pmatrix} F & 0 & D \\ 0 & -3E & B \\ D & B & C \end{pmatrix} \end{aligned} \tag{5.4a}$$

with only D complex and the following relations holding

$$\begin{aligned} C'/C &= v_u/v_d, \qquad & E'/E &= w_u/w_d \\ f_{11}^{(10')}v'_d &= -f_{11}^{(126)}w_d = \tfrac{1}{4}F, \qquad & f_{11}^{(126)}w_u &= F' \\ f_{12}^{(10')}v'_d &= 3f_{11}^{(126')}w'_d = \tfrac{3}{4}A \end{aligned} \tag{5.4b}$$

from which we obtain the constraint, $4F'/F = -E'/E$.

With

$$F' = -\bar{m}_u, \qquad E' = \bar{m}_c, \qquad C' = \bar{m}_t \tag{5.5a}$$

$$
\begin{aligned}
C &= 2.4607, & &\text{so} \quad v_u/v_d = \tan\beta = 48.9 \\
E &= -0.3830 \times 10^{-1}, & &\text{hence} \quad w_u/w_d = -8.20 \\
F &= -0.5357 \times 10^{-3}, & B &= 0.8500 \times 10^{-1} \\
A &= -0.9700 \times 10^{-2}, & D &= (0.4200 + 0.4285i) \times 10^{-2}
\end{aligned}
\tag{5.5b}
$$

the masses and mixing matrices are calculated at the GUT scale by use of the projection operator technique of Jarlskog[24] and then evolved to the low scales. The following low-scale results emerge for the quarks:

$$
\begin{aligned}
m_u(1\mathrm{GeV}) &= 5.10\ \mathrm{MeV}, & m_d(1\mathrm{GeV}) &= 9.33\ \mathrm{MeV} \\
m_c(m_c) &= 1.27\ \mathrm{GeV}, & m_s(1\mathrm{GeV}) &= 181\ \mathrm{MeV} \\
m_t(m_t) &= 150\ \mathrm{GeV}, & m_b(m_b) &= 4.09\ \mathrm{GeV}
\end{aligned}
\tag{5.6a}
$$

$$
V_{CKM} = \begin{pmatrix} 0.9753 & 0.2210 & (0.2089 - 0.2242i) \times 10^{-2} \\ -0.2209 & 0.9747 & 0.0444 \\ 0.0078 - 0.0022i & -0.0438 - 0.0005i & 0.9994 \end{pmatrix} \tag{5.6b}
$$

These results are in excellent agreement with the input in (2.1a,b), aside from the unknown CP phase, with $|V_{ub}/V_{cb}| = 0.069$ and $m_s/m_d = 19.4$, cf. Refs. 11, 12 and 15.

In the absence of any **126** VEV coupling the left-handed neutrino fields together, we observe that the heavy righthanded Majorana neutrino mass matrix can be computed at the GUT scale from the approximate seesaw mass formula[25]

$$
M^R = -M^{N_{Dirac}}(M^{N_{eff}})^{-1}M^{N_{Dirac}} \tag{5.7}
$$

$$
= \begin{pmatrix} (.1744 - .0044i) \times 10^{10} & (-.2332 + .0153i) \times 10^{11} & (-.2811 - .1925i) \times 10^{12} \\ (-.2332 + .0153i) \times 10^{11} & (.6773 - .0329i) \times 10^{12} & (-.1189 + .0243i) \times 10^{14} \\ (-.2811 - .1925i) \times 10^{12} & (-.1189 + .0243i) \times 10^{14} & (.6045 + .0624i) \times 10^{15} \end{pmatrix}
$$

by making use of Eqs. (5.4a) and (5.5a) for the Dirac neutrino matrix and (4.5d) for the effective light neutrino matix. Numerically this can be well approximated by the nearly geometric form

$$M^R = \begin{pmatrix} F'' & -\frac{2}{3}\sqrt{F''E''} & -\frac{1}{3}\sqrt{F''C''}e^{i\phi_{D''}} \\ -\frac{2}{3}\sqrt{F''E''} & E'' & -\frac{2}{3}\sqrt{E''C''}e^{i\phi_{B''}} \\ -\frac{1}{3}\sqrt{F''C''}e^{i\phi_{D''}} & -\frac{2}{3}\sqrt{E''C''}e^{i\phi_{B''}} & C'' \end{pmatrix} \qquad (5.8a)$$

where $E'' = \frac{2}{3}\sqrt{F''C''}$ and $\phi_{B''} = -\phi_{D''}/3$. With $C'' = 0.6077 \times 10^{15}$, $F'' = 0.1745 \times 10^{10}$ and $\phi_{D''} = 45^o$, we find numerically

$$M^R = \begin{pmatrix} 0.1745 \times 10^{10} & -0.2307 \times 10^{11} & (-.2427 - .2427i) \times 10^{12} \\ -0.2307 \times 10^{11} & 0.6865 \times 10^{12} & (-.1315 + .0352i) \times 10^{14} \\ (-.2427 - .2427i) \times 10^{12} & (-.1315 + .0352i) \times 10^{14} & 0.6077 \times 10^{15} \end{pmatrix} \qquad (5.8b)$$

The structure in (5.8a) can be separated into two parts with coefficients 2/3 and 1/3 which suggests they may arise again from two different **126** contributions. Such geometric textures have been studied at some length by Lemke[26] and provide a new mechanism for leptogenesis as suggested by Murayama.[27] The resulting heavy Majorana neutrino masses are found from (5.7b) to be

$$\begin{aligned} M_{R_1} &= 0.249 \times 10^9 \text{ GeV} \\ M_{R_2} &= 0.451 \times 10^{12} \text{ GeV} \\ M_{R_3} &= 0.608 \times 10^{15} \text{ GeV} \end{aligned} \qquad (5.8c)$$

By making use of the simplified matrices at the GUT scale first to compute the lepton masses and mixing matrix V_{LEPT} again by the projection operator technique of Jarlskog[24] and then to evolve the results to the low scales, we find at the low scales

$$\begin{aligned} m_{\nu_e} &= 0.534 \times 10^{-5} \text{ eV}, & m_e &= 0.504 \text{ MeV} \\ m_{\nu_\mu} &= 0.181 \times 10^{-2} \text{ eV}, & m_\mu &= 105.2 \text{ MeV} \\ m_{\nu_\tau} &= 0.135 \text{ eV}, & m_\tau &= 1.777 \text{ GeV} \end{aligned} \qquad (5.9a)$$

and

$$V_{LEPT} = \begin{pmatrix} 0.9990 & 0.0451 & (-0.029 - 0.227i) \times 10^{-2} \\ -0.0422 & 0.9361 & 0.3803 \\ 0.0174 - 0.0024i & -0.3799 - 0.0001i & 0.9371 \end{pmatrix} \tag{5.9b}$$

The agreement with our starting input is remarkably good, aside from the CP-violating phases, especially since only 12 model parameters have been introduced in order to explain 15 masses and 8 effective mixing parameters. Although we need two **10** and two **126** Higgs representations for the up, down, charged lepton and Dirac neutrino matrices with one or two additional **126**'s for the Majorana matrix, pairs of irreducible representations more naturally emerge in the superstring framework than do single Higgs representations. We have thus demonstrated by the model constructed that all quark and lepton mass and mixing data (as assumed herein) can be well understood in the framework of a simple SUSY GUT model based on SO(10) symmetry.

B. Neutrino Scenario (B) with a 7 eV Tau-Neutrino

We now apply the same type of reasoning as above to construct an SO(10) model incorporating a 7 eV tau-neutrino with small mixing with the muon-neutrino as suggested in the cocktail model[6] interpretation of mixed dark matter. As noted earlier in Sect. IV, we shall pursue the analysis with the choice of $x_q = 0$ and $x_\ell = 0.3$ which leads to four texture zeros in the quark mass matrices rather than the simpler SO(10) construct with $x_q = 0.5$ and $x_\ell = 0$, leading to only one texture zero.

Analysis of the down and charged lepton mass matrices in (4.5b) and (4.6a) with the help of (4.4b) reveals that the observed structure for M^D and M^E is now approximately given by

$$M^D \sim M^E \sim \begin{pmatrix} 10,126 & 10,126 & 10 \\ 10,126 & 126 & 10,126 \\ 10 & 10,126 & 10 \end{pmatrix} \tag{5.10}$$

with M^D_{11} again anomalously small. In fact, since $x_q = 0$ is the same in both scenarios, the quark mass matrix textures are just M^U and M^D in (5.4) with the same choice of parameters as listed in (5.5a,b), since we must fit the same low scale quark masses and CKM mixing matrix. Only M^E and $M^{N_{Dirac}}$ differ and are modified as given below.

$$\begin{aligned} M^U &= diag(F',\ E',\ C') & M^{N_{Dirac}} &= diag(-2.5F',\ -3E',\ C') \\ M^D &= \begin{pmatrix} 0 & A & D \\ A & E & B \\ D & B & C \end{pmatrix} & M^E &= \begin{pmatrix} \frac{5}{6}F & -\frac{1}{3}A & D \\ -\frac{1}{3}A & -3E & \frac{1}{3}B \\ D & \frac{1}{3}B & C \end{pmatrix} \end{aligned} \tag{5.11}$$

with only D again complex.

The heavy righthanded Majorana neutrino mass matrix can again be computed at the GUT scale from the approximate seesaw mass formula (5.7), and we find with the help of $M^{N_{Dirac}}$ in (5.11) and $M^{N_{eff}}$ in (4.6b)

$$M^R = \begin{pmatrix} (-.1096 + .0059i) \times 10^{11} & (.5523 - .0151i) \times 10^{11} & (.4594 + .1745i) \times 10^{12} \\ (.5523 - .0151i) \times 10^{11} & (.6315 + .0026i) \times 10^{11} & (-.2478 - .0939i) \times 10^{13} \\ (.4594 + .1745i) \times 10^{12} & (-.2478 + .0939i) \times 10^{13} & (-.1733 - .1547i) \times 10^{14} \end{pmatrix} \tag{5.12}$$

Numerically this can again be well approximated by the nearly geometric form

$$M^R = \begin{pmatrix} F'' & -2\sqrt{F''E''} & -\sqrt{F''C''}e^{i\phi_{D''}} \\ -2\sqrt{F''E''} & -E'' & 2\sqrt{E''C''}e^{i\phi_{B''}} \\ -\sqrt{F''C''}e^{i\phi_{D''}} & 2\sqrt{E''C''}e^{i\phi_{B''}} & C'' \end{pmatrix} \tag{5.13a}$$

where $E'' = \frac{1}{8}\sqrt{F''C''}$, aside from an overall sign. With $C'' = 0.2323 \times 10^{14}$, $F'' = 0.1096 \times$

10^{11} and $\phi_{D''} = 18.7^o$, $\phi_{B''} = 23.0^o$ and $\phi_{C''} = 41.8^o$, we find numerically

$$M^R = \begin{pmatrix} -0.1096 \times 10^{11} & 0.5258 \times 10^{11} & (.4779 + .1618i) \times 10^{12} \\ 0.5258 \times 10^{11} & 0.6307 \times 10^{11} & (-.2228 + .0946i) \times 10^{13} \\ (.4779 + .1618i) \times 10^{12} & (-.2228 - .0946i) \times 10^{13} & (-.1732 - .1548i) \times 10^{14} \end{pmatrix} \tag{5.13b}$$

By means of a phase transformation, one can reduce the three phases to two without changing the physical content of the mass and mixing matrices.[23] Again the structure in (5.13a) can be separated into two parts now with equal coefficients which suggests they may arise from two different **126** contributions. The resulting heavy Majorana neutrino masses for this case are found from (5.13b) to be

$$\begin{aligned} M_{R_1} &= 0.841 \times 10^9 \text{ GeV} \\ M_{R_2} &= 0.312 \times 10^{12} \text{ GeV} \\ M_{R_3} &= 0.235 \times 10^{14} \text{ GeV} \end{aligned} \tag{5.13c}$$

By again making use of the simplified matrices at the GUT scale first to compute the lepton masses and mixing matrix V_{LEPT} by the projection operator technique of Jarlskog[24] and then to evolve the results to the low scales, we find at the low scales for the (B) scenario

$$\begin{aligned} m_{\nu_e} &= 0.544 \times 10^{-6} \text{ eV}, & m_e &= 0.511 \text{ MeV} \\ m_{\nu_\mu} &= 0.242 \times 10^{-2} \text{ eV}, & m_\mu &= 107.9 \text{ MeV} \\ m_{\nu_\tau} &= 6.99 \text{ eV}, & m_\tau &= 1.776 \text{ GeV} \end{aligned} \tag{5.14a}$$

and

$$V_{LEPT} = \begin{pmatrix} 0.9992 & 0.0410 & (0.150 - 0.107i) \times 10^{-2} \\ -0.0411 & 0.9991 & 0.0113 \\ -0.0010 - 0.0011i & -0.0123 & 0.9999 \end{pmatrix} \tag{5.14b}$$

VI. SUMMARY

In this paper we have demonstrated how one can apply the procedure outlined in the introduction to two different neutrino scenarios to construct model mass matrices which fit well the low energy input data assumed at the outset. The full set of quark, lepton and neutrino masses and their mixing matrices are evolved to the grand unification scale, where the mass matrices can be constructed numerically by an extension of Kusenko's method which he applied only to quarks. A key ingredient is the possibility to vary two parameters, x_q and x_l, as well as the signs of the masses in the diagonal matrices, which allow one to scan the bases for a choice where the mass matrices may exhibit simple SO(10) structure. Knowledge of the preferred bases and SO(10) symmetry structure of the up, down, charged lepton and Dirac neutrino mass matrices then allows one to construct a set of model matrices with a small set of parameters. From the numerical light neutrino and Dirac neutrino mass matrices, one can then deduce the structure of the heavy right-handed Majorana mass matrix.

In scenario (A) where one makes use of data on the nonadiabatic MSW interpretation of the solar neutrino flux depletion as well as the observed depletion of atmospheric muon-neutrinos, we identified a simple SO(10) structure for the selection of $x_q = 0$ and $x_\ell = 0.88$. In this basis, the up quark matrix is real and diagonal, while the down quark, charged lepton and light neutrino mass matrices are complex symmetric. Not only is the symmetry structure simple, but the maximum number of texture zeros is obtained for the mass matrices. In particular, we find that a minimum of two **10**'s and two **126**'s of Higgs representations are required for the mass generation, while only four texture zeros appear in the up and down quark mass matrices. This should be contrasted with other authors' assumptions[4,5] of just one set of **10**'s and **126**'s or a minimum of five texture zeros.

In scenario (B), data on the nonadiabatic MSW solar depletion effect is used together with that for a 7 eV tau-neutrino which provides the hot dark matter component of mixed dark matter. In this case the choice of $x_q = 0.5$ and $x_\ell = 0$ provides the simplest SO(10) structure, but only one texture zero appears. As an alternative, we have sacrificed some structure simplicity to maintain four texture zeros in the up and down quark matrices by selecting instead $x_q = 0$ and $x_\ell = 0.3$. The up and down quark mass matrices then have exactly the same structure as in scenario (A), for the same quark masses and CKM mixing matrix must be obtained as before, but the charged lepton and neutrino matrices are now different. In view of the goal of constructing quark and lepton mass matrices which exhibit the simplest SO(10) structure and the largest number of texture zeros, we conclude that scenario (A) is favored over that for scenario (B).

We have also explored the sensitivity of our results to some changes in the input parameters. In particular, we find that the results obtained change little if one varies the electron-neutrino mass from 10^{-4} eV to 10^{-10} eV. The only pronounced effect is a change in the heavy right-handed Majorana mass matrix and its eigenvalues which can shift by one order of magnitude for the range of m_{ν_e} considered. Likewise, we find that if one lowers V_{cb} to 0.038, the simple SO(10) structure is retained in scenario (A), for example, with $x_q = 0$ and $x_\ell = 0.90$; similarly for scenario (B). Small changes in the model parameters occurring in (5.5) then again permit good reproduction of the initial input data.

Work is now underway to try to identify discrete symmetries or family symmetries which will lead to the matrix models constructed at the GUT scale. We also are studying a similar type of analysis for higher symmetry groups such as SO(18).

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